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COHOMOLOGY OF DISCRETE GROUPS IN HARMONIC COCHAINS ON BUILDINGS

BY

GIL ALON AND EHUD DE SHALIT

Institute of Mathematics, The Hebrew University of Jerusalem
Givat Ram, Jerusalem 91904, Israel
e-mail: alongil@math.huji.ac.il, deshalit@math.huji.ac.il

ABSTRACT

Modules of harmonic cochains on the Bruhat–Tits building of the projective general linear group over a p-adic field were defined by one of the authors, and were shown to represent the cohomology of Drinfel'd's p-adic symmetric domain. Here we define certain non-trivial natural extensions of these modules and study their properties. In particular, for a quotient of Drinfel'd's space by a discrete cocompact group, we are able to define maps between consecutive graded pieces of its de Rham cohomology, which we show to be (essentially) isomorphisms. We believe that these maps are graded versions of the Hyodo–Kato monodromy operator N.

1. Introduction

Let K be a finite extension of $\mathbb{Q}_p$, and $\mathcal{T}$ the tree of $G = PGL_2(K)$, a $(q+1)$-regular tree on which G acts. Let M be an abelian group, and let C^k $(k = 0, 1)$ denote the group of alternating, M-valued, k-cochains on $\mathcal{T}$. As usual, the coboundary map $d: C^0 \to C^1$ is defined by

$$df(uv) = f(v) - f(u).$$

The subgroup of **harmonic cochains** $C^k_{har} \subset C^k$ is defined as follows. In degree 0, $C^0_{har} = M$ are simply the constant maps from the vertices of $\mathcal{T}$ to M. In degree 1, C^1_{har} consists of all the maps f from the oriented edges to M, which satisfy, for every vertex v,

$$\sum_u f(uv) = 0$$

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(the sum is over the vertices adjacent to v), and of course, $f(uv) = -f(vu)$. Define also the subgroup $\tilde{C}^0_{har} \subset C^0$ to consist of all the maps g from the vertices of $\mathcal{T}$ to M satisfying, for every vertex v,

$$\sum_u (g(v) - g(u)) = 0.$$

Since $\mathcal{T}$ is simply connected, the following sequence of G-modules is exact:

$$0 \to C^0_{har} \to \tilde{C}^0_{har} \xrightarrow{d} C^1_{har} \to 0.$$

Its importance stems from the fact that, when $M = K$, C^k_{har} is canonically identified (via the **residue map**) with the kth de Rham cohomology of the Drinfel'd p-adic "upper half plane" $\mathfrak{X}$, which lurks behind the purely combinatorial picture of the tree. Similarly we could use l-adic coefficients to identify C^k_{har} with the rigid-étale l-adic cohomology. The existence of a canonical extension of C^1_{har} by C^0_{har} is a special phenomenon, peculiar to the cohomology of spaces like $\mathfrak{X}$, which admit a "totally degenerate reduction" modulo p. From a purely representation-theoretic point of view, C^1_{har} is the (full algebraic) dual of the Steinberg representation of G, and the extension in question is of course well-known. Thus what is interesting here is the way this extension is modeled on harmonic cochains on the tree, and the way it is related to the cohomology of $\mathfrak{X}$.

In [dS] and [A-dS] we defined d-dimensional analogues of these, when $\mathcal{T}$ is now the Bruhat–Tits building of $PGL_{d+1}(K)$. We defined spaces $C^k_{har} \subset C^k$ characterized by "harmonicity conditions" for every $0 \leq k \leq d$ (see 2.5 ahead), proved that via appropriate **residue maps** they become isomorphic to the kth cohomology of Drinfel'd's p-adic symmetric domain $\mathfrak{X}$ of dimension d, and established the relation between our description of that cohomology, and other descriptions due to Schneider and Stuhler [S-S] and to Iovita and Spiess [I-S].

The first goal of this work is to construct canonical extensions of G-modules

$$0 \to C^{k-1}_{har} \to \tilde{C}^{k-1}_{har} \xrightarrow{d} C^k_{har} \to 0$$

for every $1 \leq k \leq d$, and to describe $\tilde{C}^{k-1}_{har}$ by means of certain harmonicity conditions on $k-1$ cochains on the building. This is achieved in Proposition 3.4 and Theorem 3.6. These short exact sequences are obtained by dualizing short exact sequences of smooth G-modules, whose construction is purely combinatorial, and somewhat tricky. As in the one-dimensional case, the extensions in question are easy to define representation-theoretically, and the chief interest is in modeling them on harmonic cochains, and relating them to the combinatorics of $\mathcal{T}$ and to the cohomology of $\mathfrak{X}$.

The second goal of this work is to examine the cohomology of a discrete, cocompact subgroup Γ of G, with values in these spaces. We assume that Γ acts trivially on M, although in future work we hope to address the case where M is an algebraic representation of Γ. When $M = K$, the groups $H^r(\Gamma, C^s_{har})$ are the graded pieces in the covering filtration (which coincides with the weight filtration) of the cohomology of the algebraic variety $X_\Gamma = \Gamma\backslash\mathfrak{X}$ (this quotient is algebraizable by a theorem of Mumford and Mustafin). They have been studied by Schneider and Stuhler, Iovita and Spiess, and also by us in [A-dS]. They are interesting only in the range $r + s = d$. Our interest here is in the connecting homomorphism

$$\nu\colon H^r(\Gamma, C^s_{har}) \to H^{r+1}(\Gamma, C^{s-1}_{har}),$$

resulting from the extension $\tilde{C}^{s-1}_{har}$. Our main result (Theorem 4.3) is that ν is an isomorphism, except for the cases $r = s$ or $r+1 = s-1$, where it is surjective or injective, with a 1-dimensional kernel or cokernel, respectively. We prove this result by showing that the dth iteration of the connecting homomorphism

$$\nu^d\colon H^0(\Gamma, C^d_{har}) \to H^d(\Gamma, K)$$

is the well-known isomorphism used by Garland in [G].

Although the cohomology of X_Γ is nowhere mentioned in our work, questions about the action of a certain **monodromy operator** on it motivate the study of ν. To explain this, assume once again that $d = 1$, so X_Γ is a Mumford curve. In this case one has two short exact sequences which fit into the diagram at the end of this paragraph. The horizontal sequence is the Hodge filtration exact sequence, and the vertical one the covering filtration exact sequence. The de Rham cohomology can be described as the space of differentials of the second kind on X_Γ, modulo exact differentials. Pulling back to $\mathfrak{X}$, any differential of the second kind can be assigned well-defined **residues** along the oriented rigid analytic open annuli which are the pre-images, under the reduction map, of the (oriented) edges of the tree $\mathcal{T}$. The **residue theorem** guarantees that the 1-cochains thus obtained are harmonic, and of course, they are also Γ-invariant. This is the map denoted by res in the diagram. Its kernel, the differentials of the second kind with vanishing residues, modulo the exact ones, is canonically identified with $H^1(\Gamma, K)$, as is most easily seen using Cech cocycles to compute the cohomology. The four vector spaces at the edges of the diagram are g-dimensional, where g is the genus of X_Γ. The diagonal arrows are isomorphisms. It is enough to prove that this is the case for the south-west corner, where it becomes the statement that a differential of the first kind on $\mathfrak{X}$ which is Γ invariant, and all of whose residues along annuli vanish, is identically 0. As a result we obtain a Hodge-like decomposition

$$H^1_{dR}(X_\Gamma) = H^0(X_\Gamma, \Omega^1) \oplus H^1(\Gamma, K).$$

All these statements have been generalized to de Rham cohomology with values in the local system Sym^n. See [dS2].

$$\begin{array}{ccccccccc}
 & & & & 0 & & & & \\
 & & & & \downarrow & & & & \\
 & & & & H^1(\Gamma, K) & & & & \\
 & & & & \downarrow & \searrow & & & \\
0 & \longrightarrow & H^0(X_\Gamma, \Omega^1) & \longrightarrow & H^1_{dR}(X_\Gamma) & \longrightarrow & H^1(X_\Gamma, \mathcal{O}) & \longrightarrow & 0 \\
 & & & \searrow & \downarrow{\scriptstyle res} & & & & \\
 & & & & H^0(\Gamma, C^1_{har}) & & & & \\
 & & & & \downarrow & & & & \\
 & & & & 0 & & & &
\end{array}$$

Now the de Rham cohomology of any curve over K with semi-stable reduction carries a **monodromy operator** N, which is nilpotent of rank 2 ($N^2 = 0$) and which is 0 if the curve has good reduction (or, more generally, if and only if its Jacobian has good reduction). In the case of X_Γ it can be shown that $\ker(N) = H^1(\Gamma, K)$, and the resulting map $\bar{N}$: $H^0(\Gamma, C^1_{har}) \to H^1(\Gamma, K)$ is just our ν. See the work by Coleman and Iovita [C-I], where they also discuss the Frobenius structure on the de Rham cohomology.

These results have been generalized to the varieties uniformized by the d-dimensional $\mathfrak{X}$, i.e. to the X_Γ's discussed above in [S-S], [S], [I-S] and [A-dS]. The two short exact sequences are replaced by two spectral sequences. The Hodge-to-de Rham spectral sequence degenerates at E_1, leading to the Hodge filtration, and the covering spectral sequence degenerates at E_2, leading to the covering filtration. The two filtrations are opposite, resulting in isomorphisms of the graded pieces

$$H^s(X_\Gamma, \Omega^r) \simeq H^s(\Gamma, C^r_{har}) = H^{r,s}$$

(generalizing the two diagonal arrows in the diagram above), and in a Hodge-like decomposition of the cohomology

$$H^k_{dR}(X_\Gamma) = \bigoplus_{r+s=k} H^{r,s}.$$

Hyodo and Kato [H] defined, for any smooth projective variety with semi-stable reduction, a monodromy operator N on its de Rham cohomology. In our case, the only interesting cohomology is $H^d_{dR}(X_\Gamma)$, on which N is nilpotent of rank $d+1$. It is not difficult to see that N respects the covering filtration, and induces 0 on the graded pieces. Looking at the "first diagonal above the main diagonal" we see that it induces maps

$$\bar{N}\colon H^s(\Gamma, C^r_{har}) \to H^{s+1}(\Gamma, C^{r-1}_{har}).$$

We conjecture that $\bar{N} = \nu$, as was proved in the 1-dimensional case by Coleman and Iovita.* If correct, this would imply, via our main theorem on ν, a conjecture of Mokrane [M] on the monodromy filtration. In the l-adic setting, this conjecture is Deligne's conjecture on the "purity of the monodromy filtration" (saying that the monodromy filtration, suitably renumbered, coincides with the weight filtration). It is known to have important consequences for the study of local L factors of Shimura varieties at bad primes. Although the varieties of type X_Γ are special among the varieties with semi-stable reduction, and Deligne's conjecture, or Mokrane's, are probably more accessible for them, we find this line of attack interesting.

The rest of this paper is organized as follows: In section 2 we gather the necessary notation and results of previous work. In section 3 we describe the extension $\tilde{C}^k_{har}$, and in section 4 we prove our main result on ν.

2. Notation and background

2.1. THE BRUHAT–TITS BUILDING. Let K be a finite extension of $\mathbb{Q}_p$, and π a uniformizer. Let V_K be a $(d+1)$-dimensional vector space, $G = PGL(V_K)$ and denote by $\mathcal{T}$ **the Bruhat–Tits building** of G. It is the simplicial complex in which the vertices are the dilation classes $[L]$ of lattices in V_K and the k-simplices are the sets $\{[L_0], [L_1], \dots, [L_k]\}$ where $L_0 \supset L_1 \supset \cdots \supset L_k \supset \pi L_0$ (strict inclusions). These inclusions determine a canonical cyclic ordering of the vertices in any simplex. We will also use the oriented version $\widehat{\mathcal{T}}$ of this building, in which the k-simplices are the sequences $\sigma = ([L_0], [L_1], \dots, [L_k])$. The vertex $[L_0]$ is called **the distinguished vertex** of σ. The cyclic group Z_{k+1} acts on $\widehat{\mathcal{T}}_k$ by cyclically permuting the vertices.

Two simplices σ and τ of $\mathcal{T}$ (or $\hat{\mathcal{T}}$) are called **adjacent** if $\sigma \cup \tau$ is a simplex of $\mathcal{T}$. We denote by $\widehat{\mathcal{T}}_k(\tau)$ the set of oriented k-simplices that are adjacent to τ. Alternatively, σ and τ are adjacent if they are both contained in a third simplex.

* *Added in proof*: This has now been established by the second author.

2.2. THE PAIRING (σ, S). Set $\mathcal{A} = \mathbb{P}(V_K)$ (viewed as a compact, totally disconnected space under the p-adic topology, equipped with a continuous G-action).

We define a pairing $\hat{\mathcal{T}}_k \times \mathcal{A}^{k+1} \to \{-1,0,1\}$ in the following way: For $\sigma = ([L_0],[L_1],\ldots,[L_k]) \in \hat{\mathcal{T}}_k$ and $S = ([a_0],[a_1],\ldots,[a_k])$, where the representatives a_i are chosen so that $a_i \in L_0 \setminus \pi L_0$, $(\sigma, S) = sgn(\pi)$ if there exists a permutation π of $\{0,1,\ldots,k\}$ so that $a_{\pi(i)} \in L_i \setminus L_{i+1}$ (where $L_{k+1} = \pi L_0$), and $(\sigma, S) = 0$ otherwise.

For example, when $d = 1$, the building $\mathcal{T}$ is the infinite $(q+1)$-regular tree, $\mathcal{A}$ is identified with the set of ends of the tree, and the pairing of an oriented edge $\sigma \in \hat{\mathcal{T}}_1$ with a pair of ends $([a_0],[a_1]) \in \mathcal{A}^2$ is 0 if the geodesic from $[a_0]$ to $[a_1]$ does not pass through σ, and otherwise it is 1 or -1 according to the direction of σ in this geodesic.

We shall view this pairing in two ways. For a field F, denote by $C^k(F)$ the space of **alternating F-cochains on $\hat{\mathcal{T}}_k$**, that is, functions $c\colon \hat{\mathcal{T}}_k \to F$ satisfying for all $\sigma \in \hat{\mathcal{T}}_k$, $z \in Z_{k+1}$, $c(z(\sigma)) = sgn(z)c(\sigma)$.

Denote also, for a topological space X, $L(X) = C^\infty(X,\mathbb{Z})$ the abelian group of **locally constant functions from X to $\mathbb{Z}$.**

Then for $S \in \mathcal{A}^{k+1}$ we have a cochain $c_S \in C^k(F)$, and for $\sigma \in \hat{\mathcal{T}}_k$ we have a function $\lambda_\sigma \in L(\mathcal{A}^{k+1})$, both defined by

$$\lambda_\sigma(S) = c_S(\sigma) = (\sigma, S).$$

The group G acts on $L(\mathcal{A}^{k+1})$ by $(g\cdot f)(S) = f(g^{-1}S)$. Note that $g\cdot\lambda_\sigma = \lambda_{g\sigma}$.

2.3. LOCAL SYSTEMS. **Let R be a commutative ring. A (co)homological local system $\underline{A}$** of R-modules on $\mathcal{T}$ consists of an R-module $A(\tau)$ for every $\tau \in \mathcal{T}$ and maps $\iota_{\tau_1}^{\tau_2}\colon A(\tau_1) \to A(\tau_2)$ (resp. $r_{\tau_2}^{\tau_1}\colon A(\tau_2) \to A(\tau_1)$) whenever $\tau_1 \geq \tau_2$, satisfying the usual compatability relations. It is called **G-equivariant** if in addition there are isomorphisms $\phi_{g,\tau}\colon A(\tau) \to A(g\tau)$ commuting with the $\iota_{\tau_1}^{\tau_2}$ (resp. $r_{\tau_2}^{\tau_1}$), and satisfying $\phi_{h,g\tau} \circ \phi_{g,\tau} = \phi_{hg,\tau}$.

If a ring homomorphism $R \to S$ is given, we have the **dual system** of S-modules defined by

$$A^*(\tau) = Hom_R(A(\tau), S).$$

The dual of a homological system is a cohomological one and vice versa.

To a (co)homological local system $\underline{A}$ we attach the differential complex $C_\bullet(\hat{\mathcal{T}},\underline{A})$ (resp. $C^\bullet(\hat{\mathcal{T}},\underline{A})$) in which $C_r(\hat{\mathcal{T}},\underline{A})$ (resp. $C^r(\hat{\mathcal{T}},\underline{A})$) for $r \geq 0$ is the collection of **finitely supported** maps (resp. maps) $f\colon \hat{\mathcal{T}}_r \to \coprod_{\tau\in\mathcal{T}_r} A(\tau)$ satisfying $f(\tau) \in A(\tau)$ and $f(z(\tau)) = sgn(z)f(\tau)$. In the cohomological case, the differential $d\colon C^r(\widehat{\mathcal{T}}, \underline{A}) \to C^{r+1}(\widehat{\mathcal{T}}, \underline{A})$ is defined by

$$df(\tau) = \sum_{\tau' \in \mathcal{T}_r, \tau' \leq \tau} [\tau : \tau'] r^{\tau}_{\tau'} f(\tau')$$

where the **incidence number** $[\tau : \tau']$ is given by

$$[(v_0, \ldots, v_{r+1}) : (v_0, \ldots, \widehat{v_i}, \ldots, v_{r+1})] = (-1)^i.$$

In the homological case, $d\colon C_{r+1}(\widehat{\mathcal{T}}, \underline{A}) \to C_r(\widehat{\mathcal{T}}, \underline{A})$ is defined by

$$df(\tau) = \sum_{\tau' \in \mathcal{T}_{r+1}, \tau \leq \tau'} [\tau' : \tau] \iota^{\tau}_{\tau'} f(\tau').$$

In both cases, τ is oriented, and the sum extends over unoriented τ's. Note that each summand does not depend on the choice of orientation for τ'. For a homological local system $\underline{A}$, $C^r(\widehat{\mathcal{T}}, \underline{A}^*)$ is naturally dual to $C_r(\widehat{\mathcal{T}}, \underline{A})$, and the two (co)boundary maps are dual to each other.

2.4. THE G-MODULE Λ_k. Let $\Lambda_k \subset L(\mathcal{A}^{k+1})$ be the span of all the functions λ_σ for $\sigma \in \widehat{\mathcal{T}}_k$ (see [A-dS], cor. 2.2 for another description of Λ_k).

We shall attach to it a local system $\underline{\Lambda}_k$ on $\mathcal{T}$.

For a vertex v of $\mathcal{T}$, let B_v be its stabilizer in G, and denote by $U_v \subset B_v \simeq PGL_{d+1}(O_K)$ the principal congruence subgroup of level π, namely, if $v = [L]$,

$$U_v = \{g \in G : ga - a \in \pi L \quad \forall a \in L\}.$$

For $\tau \in \mathcal{T}_r$, let U_τ be the subgroup generated in G by all the subgroups U_v for a vertex v of τ. Define $\Lambda_k(\tau) = \Lambda_k^{U_\tau}$. It is a G-equivariant homological local system of abelian groups on $\mathcal{T}$, where the maps $\iota^{\tau_2}_{\tau_1}$ are simply the inclusion maps $\Lambda_k(\tau_1) \hookrightarrow \Lambda_k(\tau_2)$.

PROPOSITION 2.1: *$\Lambda_k(\tau)$ is spanned by the λ_σ for $\sigma \in \widehat{\mathcal{T}}_k(\tau)$.*

Proof: It is easy to see that the λ_σ for $\sigma \in \widehat{\mathcal{T}}_k(\tau)$ are fixed by U_τ. For the other direction, we use the inclusions

$$\begin{array}{ccc}
\Lambda_k^{U_\tau} & \hookrightarrow & (K \cdot \Lambda_k)^{U_\tau} \\
\big\downarrow & & \big\downarrow \\
L(\mathcal{A}^{k+1}, \mathbb{Z}) & \hookrightarrow & L(\mathcal{A}^{k+1}, K).
\end{array}$$

First we claim that $(K \cdot \Lambda_k)^{U_\tau}$ is spanned, as a K-vector space, by these λ_σ. For the proof of this fact we refer to [dS], where in (5.10) it is shown that

$(K \cdot \Lambda_k)^{U_\tau} \cong A^k(\tau)^*$ for some finite dimensional vector space $A^k(\tau)$, and corollary (2.7) gives a basis $\{\lambda_{\sigma_T}\}$ of $A^k(\tau)^*$ consisting of some of the λ_σ for $\sigma \in \widehat{\mathcal{T}}_k(\tau)$. This proves the claim, but furthermore, the elements of this basis are indexed by a subset $\mathcal{S}_{k+1}(\tau)''$ of $\mathcal{A}^{k+1}$ which is linearly ordered, and the matrix

$$(\lambda_{\sigma_T}(S))_{S,T \in \mathcal{S}_{k+1}(\tau)''}$$

is triangular, $\mathbb{Z}$-valued, with 1's on the main diagonal.

Now look at an element $\lambda \in \Lambda_k^{U_\tau}$. By the above claim, we can write

$$\lambda = \sum_{T \in \mathcal{S}_{k+1}(\tau)''} c_T \lambda_{\sigma_T}$$

with $c_T \in K$. But since $\lambda(S) \in \mathbb{Z}$ for every $S \in \mathcal{S}_{k+1}(\tau)''$, we get by backwards substitution that $c_T \in \mathbb{Z}$ for all T. ∎

We now form the chain complex $C_\bullet(\widehat{\mathcal{T}}, \underline{\Lambda_k})$ with the augmentation map

$$\epsilon\colon C_0(\widehat{\mathcal{T}}, \underline{\Lambda_k}) \to \Lambda_k$$

defined by $\epsilon(f) = \sum_{v \in \hat{\mathcal{T}}_0} f(v)$.

THEOREM 2.2: *$C_\bullet(\widehat{\mathcal{T}}, \underline{\Lambda_k}) \to \Lambda_k \to 0$ is a projective resolution of Λ_k in the category of smooth G-modules.*

Proof: See [S-S], theorem 8, p. 118. The identification of $V_I(\mathbb{Z})$ with our Λ_k is given in [A-dS], cor. 2.2. See also ibid., (2.7) and (2.11). ∎

2.5. HARMONIC COCHAINS. For a field F, we let

$$C^k_{har} = C^k_{har}(F) = Hom(\Lambda_k, F),$$

the space of **harmonic k-cochains**.

An element ϕ of this space will be viewed as an alternating cochain on $\widehat{\mathcal{T}}_k$ by defining, for $\sigma \in \hat{\mathcal{T}}_k$,

$$\phi(\sigma) = \phi(\lambda_\sigma).$$

Thus, $C^k_{har} \subset C^k(F)$. Note that $c_S \in C^k_{har}$ is the linear functional of evaluation at S.

For lattices $M \supset L \supset \pi M$ we define the **relative degree**

$$[M : L] = dim_{O_K/\pi O_K}(M/L).$$

THEOREM 2.3 ([dS], cor. 5.8): *C^k_{har} is the space of functions $c\colon \widehat{\mathcal{T}}_k \to F$ satisfying:*

(A) *For $\sigma \in \widehat{\mathcal{T}}_k$, $z \in Z_{k+1}$, $c(z(\sigma)) = sgn(z)c(\sigma)$.*

(B) *For $\sigma = (v_0, \dots, v_{k-1}) \in \widehat{\mathcal{T}}_{k-1}$, $0 \le i \le k-1$ and $0 < l < [L_i : L_{i+1}]$ ($v_i = [L_i]$),*

$$\sum_{L_i \supset M \supset L_{i+1}, [L_i:M]=l} c(v_0, \dots, v_i, [M], v_{i+1}, \dots, v_{k-1}) = 0.$$

(C) *For $\sigma = (v_0, \dots, v_k) \in \widehat{\mathcal{T}}_k$ and $1 \le i \le k$,*

$$\sum_{L_i \supseteq M \supset L_{i+1}, [M:L_{i+1}]=1} c(v_0, \dots, v_{i-1}, [M], v_{i+1}, \dots, v_k) = c(\sigma).$$

(D) *For $\sigma = (v_0, \dots, v_{k+1}) \in \widehat{\mathcal{T}}_{k+1}$,*

$$\sum_{i=0}^{k+1} (-1)^i c(v_0, \dots, \widehat{v_i}, \dots, v_{k+1}) = 0.$$

We record for future reference the following identities:

$$\sum_{i=0}^{k+1} (-1)^i \lambda_{(v_0, \dots, \widehat{v_i}, \dots, v_{k+1})} = 0 \tag{1}$$

and

$$\sum_{i=0}^{k+1} (-1)^i c_{(a_0, \dots, \hat{a}_i, \dots, a_{k+1})} = 0. \tag{2}$$

The first one is merely an expression of condition (D) for the harmonic cochain c_S. The second one is proved in [dS], 2.15.

We form the cohomological local system $\underline{A}^k = Hom(\underline{\Lambda}_k, F)$. We get a resolution of G-modules

$$0 \to C^k_{har} \to C^\bullet(\hat{\mathcal{T}}, \underline{A}^k)$$

(see [dS], corollary 5.2).

Remark 2.4: By Proposition 2.1, an element of $A^k(\tau) = \Lambda_k(\tau)^*$ is uniquely determined by its value on the elements λ_σ for $\sigma \in \hat{\mathcal{T}}_k(\tau)$. Thus, an element of $C^r(\hat{\mathcal{T}}, \underline{A}^k)$ is given by a function of two variables $f(\tau;\sigma)$ from $\{(\tau,\sigma) : \tau \in \hat{\mathcal{T}}_r, \sigma \in \hat{\mathcal{T}}_k(\tau)\}$ to F, satisfying $f(z_1\tau; z_2\sigma) = sgn(z_1)sgn(z_2)f(\tau;\sigma)$. However, not every such function represents an element of $C^r(\hat{\mathcal{T}}, \underline{A}^k)$, since the λ_σ for a given τ are linearly dependent.

3. The extension $\tilde{C}^k_{har}$

The aim of this section is to describe an extension of G-modules

$$0 \to C^{k-1}_{har} \to \tilde{C}^{k-1}_{har} \to C^k_{har} \to 0.$$

We will do it by constructing an extension of smooth G-modules

$$0 \to \Lambda_k \to \widetilde{\Lambda}_{k-1} \to \Lambda_{k-1} \to 0,$$

and dualizing it.

3.1. The functions $\widetilde{\lambda}_\sigma$.

Definition 3.1: 1. Let $\widetilde{\mathcal{A}} = (V_K \setminus \{0\})/O^*_K$. (We will slightly abuse the notation by writing $\tilde{a} \in L$ for $\tilde{a} \in \widetilde{\mathcal{A}}$ and a lattice L, since this does not depend on the representative in V_K.)

2. For a lattice L and $\tilde{a} \in \widetilde{\mathcal{A}}$, $ord_L(\tilde{a}) = n$ if $\tilde{a} \in \pi^n L \setminus \pi^{n+1} L$.
3. For $\sigma = ([L_0], \ldots, [L_{k-1}]) \in \widehat{\mathcal{T}}_{k-1}$, $\widetilde{S} = (\tilde{a}_0, \ldots, \tilde{a}_k) \in \widetilde{\mathcal{A}}^{k+1}$,

$$\widetilde{\lambda}_\sigma(\widetilde{S}) = -\sum_{i=0}^{k} (-1)^i (\sigma, S_i) ord_{L_0}(\widetilde{a_i})$$

where a_i is the image of $\widetilde{a_i}$ in $\mathcal{A}$, and $S_i = (a_0, \ldots, \widehat{a_i}, \ldots, a_k)$.

To see that this is well-defined, note that if we change the lattice-flag representing σ by a homothety, then all the $ord_{L_0}(\tilde{a}_j)$ are increased by a fixed amount, but $\sum_{j=0}^k (-1)^j (\sigma, S_j) = 0$ (see (2)). Clearly, $\tilde{\lambda}_\sigma \in L(\widetilde{\mathcal{A}}^{k+1})$.

Let $\tilde{\Lambda}_{k-1}$ be the abelian group spanned by the $\tilde{\lambda}_\sigma$ for all $\sigma \in \widehat{\mathcal{T}}_{k-1}$, and let ∂ be the inclusion $L(\mathcal{A}^{k+1}) \hookrightarrow L(\widetilde{\mathcal{A}}^{k+1})$ coming from the projection $\widetilde{\mathcal{A}} \to \mathcal{A}$. The next lemma proves that $\partial(\Lambda_k) \subset \tilde{\Lambda}_{k-1}$, and also explains why we chose to denote it by ∂.

Lemma 3.2: *One has the following formula,*

$$\partial \lambda_\sigma = \sum_{l=0}^{k} (-1)^l \tilde{\lambda}_{\sigma_l} \tag{3}$$

where, if $\sigma = (v_0, \ldots, v_k)$, $\sigma_l = (v_0, \ldots, \widehat{v_l}, \ldots, v_k)$.

Proof: Let $\sigma = ([L_0], \ldots, [L_k])$. Fix $S = (a_0, \ldots, a_k) \in \mathcal{A}^{k+1}$ and a lifting $\tilde{S}$. We have to show that

$$\lambda_\sigma(S) = \sum_{l=0}^{k} (-1)^l \tilde{\lambda}_{\sigma_l}(\tilde{S}).$$

We first develop the right hand side (RHS) using the identity (1):

$$\begin{aligned}\text{RHS} &= -\sum_{i=0}^{k}(-1)^i(\sigma_0, S_i)ord_{L_1}(\tilde{a}_i) - \sum_{l=1}^{k}\sum_{i=0}^{k}(-1)^{i+l}(\sigma_l, S_i)ord_{L_0}(\tilde{a}_i)\\ &= \sum_{i=0}^{k}(-1)^i(\sigma_0, S_i)(ord_{L_0} - ord_{L_1})(\tilde{a}_i).\end{aligned}$$

Thus it remains to prove the identity

$$(\sigma, S) = \sum_{i=0}^{k}(-1)^i(\sigma_0, S_i)(ord_{L_0} - ord_{L_1})(\tilde{a}_i). \tag{4}$$

Clearly, the RHS is independent of the lifting $\tilde{S}$. We choose $\tilde{S}$ to be **adapted to** σ, i.e. all $\tilde{a}_i \in L_0 \setminus \pi L_0$. Then $(ord_{L_0} - ord_{L_1})(\tilde{a}_i) = 1$ if $\tilde{a}_i \in L_0 \setminus L_1$ and is 0 if $\tilde{a}_i \in L_1 \setminus \pi L_0$. We distinguish cases:

- If no $\tilde{a}_i$ lies in $L_0 \setminus L_1$, both sides are clearly 0.
- If there are at least 3 such $\tilde{a}_i$, then in every S_j there are at least 2 $\tilde{a}_i$'s with the same σ_0-index, so the RHS is again 0, and so is the LHS.
- If there are precisely 2 such, assume that they are $\tilde{a}_j$ and $\tilde{a}_i$. Then the RHS is $(-1)^j(\sigma_0, S_j)+(-1)^i(\sigma_0, S_i)$. Now S_j contains a_i but not a_j, and S_i contains a_j but not a_i. From the point of view of the permutation defining (σ_0, S_i) or (σ_0, S_j), a_i and a_j have the same σ_0-index, because they belong to $\pi^{-1}L_k \setminus L_1$. Thus we find that $(\sigma_0, S_i) = (\sigma_0, S_j)(-1)^{i-j-1}$, and this proves that the RHS vanishes.
- There remains the case where there is a unique $\tilde{a}_j \in L_0 \setminus L_1$. The RHS is equal then to $(-1)^j(\sigma_0, S_j)$. Exchanging a_j and a_0 in S changes the LHS by $(-1)^j$, so we may assume that $j = 0$, i.e. $\tilde{a}_0 \in L_0 \setminus L_1$. In that case we have to prove that $(\sigma, S) = (\sigma_0, S_0)$, which clearly holds. ∎

Next, we define a map $\psi\colon \widetilde{\Lambda}_{k-1} \to \Lambda_{k-1}$. Fix an auxiliary $\tilde{a}_0 \in \widetilde{\mathcal{A}}$, and consider the map

$$\gamma_{\tilde{a}_0}\colon \widetilde{\mathcal{A}}^k \to \widetilde{\mathcal{A}}^{k+1}, \gamma_{\tilde{a}_0}(\tilde{a}_1, \ldots, \tilde{a}_k) = (\tilde{a}_0, \tilde{a}_1, \ldots, \tilde{a}_k). \tag{5}$$

Let $\gamma_{\tilde{a}_0}^*\colon L(\widetilde{\mathcal{A}}^{k+1}) \to L(\widetilde{\mathcal{A}}^k)$ be the pull-back of functions. Consider the map $\psi = \gamma_{\tilde{a}_0}^* - \gamma_{\pi\tilde{a}_0}^*$. Since $\gamma_{\pi\tilde{a}_0}^*$ and $\gamma_{\tilde{a}_0}^*$ clearly agree on $L(\mathcal{A}^{k+1})$, we see that $\psi \circ \partial = 0$. The following lemma shows that on $\widetilde{\Lambda}_{k-1}$, ψ is independent of the choice of $\tilde{a}_0$, and that its image lies in $\Lambda_{k-1} \subset L(\mathcal{A}^k) \subset L(\widetilde{\mathcal{A}}^k)$. At the same time it proves that ψ is surjective.

LEMMA 3.3: *We have*

$$\psi(\tilde{\lambda}_\tau) = \lambda_\tau. \tag{6}$$

Proof: Fix $S \in \mathcal{A}^k$ and a lifting $\tilde{S} = (\tilde{a}_1, \dots, \tilde{a}_k) \in \tilde{\mathcal{A}}^k$. Then

$$\begin{aligned}\psi(\tilde{\lambda}_\tau)(\tilde{S}) =& \tilde{\lambda}_\tau(\tilde{a}_0, \tilde{S}) - \tilde{\lambda}_\tau(\pi\tilde{a}_0, \tilde{S}) \\ =& (\tau, S) = \lambda_\tau(S). \end{aligned} \tag{7}$$ ∎

PROPOSITION 3.4: *The sequence*

$$0 \to \Lambda_k \xrightarrow{\partial} \tilde{\Lambda}_{k-1} \xrightarrow{\psi} \Lambda_{k-1} \to 0 \tag{8}$$

is exact.

Proof: It remains to prove that anything in $\ker(\psi)$ is in $\mathrm{Im}(\partial)$. Suppose that $\sum n_\tau[\tau]$ is a $(k-1)$-chain such that $\tilde{\lambda} = \sum n_\tau \tilde{\lambda}_\tau \in \ker(\psi)$. This means that $\sum n_\tau \lambda_\tau = 0$. We know that Λ_{k-1} is generated by the simplices $[\tau]$ modulo the relations of types (A)–(D). There are therefore finitely many steps of type (A)–(D) that transform the chain $\sum n_\tau[\tau]$ to 0. Now consider the effect of each such step on the corresponding $\tilde{\lambda}$. If it is a step of type (A)–(C), $\tilde{\lambda}$ is not changed. This is proved in Theorem 3.5 below. If it is a step of type (D), we add to our chain a boundary of a k-chain, so we modify $\tilde{\lambda}$ by an element coming from $\partial(\Lambda_k)$. This proves that $\tilde{\lambda} \in \mathrm{Im}(\partial)$, so the sequence is exact. ∎

It is also clear that the natural $GL(V_K)$ action on $\tilde{\Lambda}_{k-1}$ factors through G, because for every constant c, and every $\tilde{S}$, $\tilde{\lambda}_\tau(c\tilde{S}) = \tilde{\lambda}_\tau(\tilde{S})$ (again by (2)). The G-module $\tilde{\Lambda}_{k-1}$ is admissible. One clearly has, for $\gamma \in G$,

$$\gamma(\tilde{\lambda}_\tau) = \tilde{\lambda}_{\gamma(\tau)}. \tag{9}$$

3.2. DEFINITION OF $\tilde{C}^{k-1}_{har}$, AND CHARACTERIZATION BY HARMONICITY CONDITIONS. As promised we define $\tilde{C}^{k-1}_{har}(F) = Hom(\tilde{\Lambda}_{k-1}, F)$. Like before, an element of it is viewed as an alternating cochain on $\widehat{\mathcal{T}}_{k-1}$ by assigning to $\sigma \in \widehat{\mathcal{T}}_{k-1}$ its value on $\tilde{\lambda}_\sigma$. We obtain the sequence

$$0 \to C^{k-1}_{har} \xrightarrow{\iota} \tilde{C}^{k-1}_{har} \xrightarrow{d} C^k_{har} \to 0, \tag{10}$$

where ι and d are the dual maps to ψ and ∂, respectively. As cochain maps, ι is simply the inclusion (by (6)) and d is the usual coboundary map (by (3)).

It is easy to see that the modules in (8) are free over $\mathbb{Z}$, hence (10) is exact.

THEOREM 3.5: *$\tilde{C}^{k-1}_{har}(F) \subset C^{k-1}(F)$ is characterized as the subspace of all cochains $f\colon \widehat{\mathcal{T}}_{k-1} \to F$ which satisfy conditions* (A), (B) *and* (C).

Proof: Let f be in $\tilde{C}^{k-1}_{har}(F)$. We have to show that f satisfies (A)–(C) (for index $k-1$). We start with a lemma.

LEMMA 3.6: *In the definition of $\tilde{\lambda}_\tau(\tilde{S})$, one can replace ord_{L_0} with ord_{L_1} (and therefore with ord_{L_i} for any L_i which is a vertex of τ).*

Proof: We have to prove that

$$\sum_{j=0}^{k}(-1)^j(\tau, S_j)(ord_{L_0}(\tilde{a}_j) - ord_{L_1}(\tilde{a}_j)) = 0. \tag{11}$$

The proof is almost identical to the proof of formula (4) (the only difference being that this time both L_0 and L_1 are in τ), and we omit it. ∎

Now continue the proof of the theorem. In verifying (A) let

$$\tau = [L_0 \supset \cdots \supset L_{k-1} \supset \pi L_0] \tag{12}$$

and

$$\tau' = [L_1 \supset \cdots \supset \pi L_0 \supset \pi L_1]. \tag{13}$$

Then

$$\tilde{\lambda}_{\tau'}(\tilde{S}) = -(-1)^{k-1}\sum_{j=0}^{k}(-1)^j(\tau, S_j)ord_{L_1}(\tilde{a}_j) \tag{14}$$

where we have used (A) for the symbol (τ, S_j). In view of the lemma this is $(-1)^{k-1}\tilde{\lambda}_\tau(\tilde{S})$.

In verifying (B) fix $\tau \in \widehat{\mathcal{T}}_{k-2}$ and let $\mathcal{C}$ be the collection of $(k-1)$-simplices appearing in the sum. Since they all have the same distinguished vertex, the equality

$$\sum_{\tau' \in \mathcal{C}} \tilde{\lambda}_{\tau'}(\tilde{S}) = 0 \tag{15}$$

follows at once from the already known harmonicity condition

$$\sum_{\tau' \in \mathcal{C}} (\tau', S_j) = 0 \tag{16}$$

after changing the summation order (first on j, then on τ').

In verifying (C) recall that the sum extends over the collection of τ' which differ from τ only in the ith lattice, and furthermore satisfy

$$L_i \supseteq L'_i \supset L_{i+1}, [L'_i : L_{i+1}] = 1, \tag{17}$$

where the index $i \geq 1$ is fixed. We have to show that $\tilde{\lambda}_\tau(\tilde{S}) = \sum_{\tau'} \tilde{\lambda}_{\tau'}(\tilde{S})$. All the τ' share the same leading vertex $v_0 = [L_0]$, and changing the summation order the property follows from the identities

$$(\tau, S_j) = \sum_{\tau'} (\tau', S_j). \tag{18}$$

This concludes the proof of one direction in the theorem. Note that this is the only direction used in the proof of the exactness of (8) above. For the other direction, we need the following lemma.

LEMMA 3.7: *Assume that* $f\colon \hat{\mathcal{T}}_{k-1} \to F$ *satisfies* (A)–(C). *Then* $df \in C^k_{har}$.

Proof: It is clear that df satisfies (A) (since f does) and (D) (since $d^2 = 0$). For condition (B), fix $\sigma = (v_0, \ldots, v_{k-1}) \in \hat{\mathcal{T}}_{k-1}$, $0 \leq i \leq k-1$ and $0 < l < [L_i : L_{i+1}]$ ($v_i = [L_i]$). Then we have to prove that

$$\sum_M df(v_0, \ldots, v_i, [M], v_{i+1}, \ldots, v_{k-1}) = 0$$

where $L_i \supset M \supset L_{i+1}$ and $[L_i : M] = l$. We will prove it for $l = [L_i : L_{i+1}] - 1$. After proving independently that df satisfies (C), it will follow that df satisfies (B) for all l. The above sum is now equal to

$$\sum_{j=0}^{i} \sum_M (-1)^j f(v_0 \ldots, \hat{v}_j, \ldots, v_i, [M], v_{i+1}, \ldots, v_{k-1}) + \sum_M (-1)^{i+1} f(v_0, \ldots, v_{k-1})$$
$$+ \sum_{j=i+1}^{k-1} \sum_M (-1)^{j+1} f(v_0, \ldots, v_i, [M], v_{i+1}, \ldots, \hat{v}_j, \ldots v_{k-1}).$$

The sums for $j < i$ and $j > i+1$ are 0 by condition (B) for f. We are left with

$$(-1)^i \sum_M (f(v_0 \ldots, v_{i-1}, [M], v_{i+1}, \ldots, v_{k-1}) - f(v_0, \ldots, v_{k-1})$$
$$+ f(v_0 \ldots, v_i, [M], v_{i+2}, \ldots, v_{k-1})).$$

The first summand adds up to $f(v_0, \ldots, v_{k-1})$, by condition (C) for f. The second one is constant, and appears as many times as the number of M's, which is the number of one dimensional subspaces of $\mathbb{F}_q^{[L_i:L_{i+1}]}$. Denote this number momentarily by n. We have to prove that

$$\sum_M f(v_0\dots,v_i,[M],v_{i+2},\dots,v_{k-1}) = (n-1)f(v_0,\dots,v_{k-1}).$$

By condition (C) for f we have

$$\sum_M f(v_0,\dots,v_i,[M],v_{i+2},\dots,v_{k-1}) = \sum_M\sum_L f(v_0,\dots,v_i,[L],v_{i+2},\dots,v_{k-1})$$

where $M \supseteq L \supset L_{i+2}$ and $[L : L_{i+2}] = 1$. We now count how many times a lattice satisfying $L_i \supset L \supset L_{i+2}$ and $[L : L_{i+2}] = 1$ appears in the double sum: If $L_{i+1} \supseteq L$ then L_{i+1} is "buffering" between L and M, hence L appears n times. If $L_{i+1} \not\supseteq L$, then L appears **exactly once**, since M is determined by $M = L + L_{i+1}$. Splitting the sum in this way gives

$$\begin{aligned}\sum_M &f(v_0,\dots,v_i,[M],v_{i+2},\dots,v_{k-1})\\ &= n\sum_{L_{i+1}\supseteq L} f(\dots,v_i,[L],v_{i+2},\dots) + \sum_{L_{i+1}\not\supseteq L} f(\dots,v_i,[L],v_{i+2},\dots)\\ &= (n-1)\sum_{L_{i+1}\supseteq L} f(\dots,v_i,[L],v_{i+2},\dots) + \sum_L f(\dots,v_i,[L],v_{i+2},\dots).\end{aligned}$$

The first summand is $(n-1)f(v_0,\dots,v_{k-1})$ by (C), and the second one vanishes by (B).

Let us now prove (C) for df. Let $\sigma = (v_0,\dots,v_k) \in \widehat{\mathcal{T}}_k$ and $1 \le i \le k$. We have to prove

$$\sum_M df(v_0,\dots,v_{i-1},[M],v_{i+1},\dots,v_k) = df(v_0,\dots,v_k),$$

where $L_i \supseteq M \supset L_{i+1}$ and $[M : L_{i+1}] = 1$. In a similar fashion to the proof of (B), after expanding both sides, using (C) for f and cancelling, we are left with proving that

$$\begin{aligned}\sum_M f(v_0\cdots v_{i-1},&v_{i+1}\cdots v_k) - f(v_0\cdots v_{i-1},[M],v_{i+2}\cdots v_k)\\ &= f(v_0\cdots\hat{v}_i\cdots v_k) - f(v_0\cdots\hat{v}_{i+1}\cdots v_k).\end{aligned} \tag{19}$$

Let again n be the number of M's in the sum, and expand

$$\sum_M f(v_0,\dots,v_{i-1},[M],v_{i+2},\dots,v_k) = \sum_M\sum_L f(v_0,\dots,v_{i-1},[L],v_{i+2},\dots,v_k)$$

where $M \supseteq L$ and $[L : L_{i+2}] = 1$. Accounting for the lattices $L_i \supseteq L \supset L_{i-2}$, every $L_{i+1} \supseteq L$ appears n times and every $L_{i+1} \not\supseteq L$ appears once. Hence the left hand side of (19) equals

$$\begin{aligned} & nf(v_0 \dots \hat{v}_i \dots v_k) - n \sum_{L_{i+1} \supseteq L} f(\dots v_{i-1}, [L], v_{i+2} \dots) \\ & \qquad - \sum_{L_{i+1} \not\supseteq L} f(\dots v_{i-1}, [L], v_{i+2} \dots) \\ &= nf(v_0 \dots \hat{v}_i \dots v_k) - nf(v_0 \dots \hat{v}_i \dots v_k) - \sum_{L} f(\dots v_{i-1}, [L], v_{i+2} \dots) \\ & \qquad + \sum_{L_{i+1} \supseteq L} f(\dots v_{i-1}, [L], v_{i+2} \dots) \\ &= -f(v_0 \dots v_{i-1}, v_i, v_{i+2} \dots v_k) + f(v_0 \dots v_{i-1}, v_{i+1}, v_{i+2} \dots v_k). \qquad \blacksquare \end{aligned}$$

We now end the proof of the theorem: Assume that f satisfies (A)–(C). By the lemma, $df \in C^k_{har}(F)$. By the surjectivity in the exact sequence (8), there is $g \in \tilde{C}^{k-1}_{har}(F)$ such that $dg = df$. It follows that $g - f$ satisfies (A)–(C) but also (D), so is in C^{k-1}_{har}. Thus $f = g - (g - f)$ is also in $\tilde{C}^{k-1}_{har}(F)$. $\blacksquare$

3.3. Local systems for $\tilde{\Lambda}_{k-1}$ and $\tilde{C}^{k-1}_{har}$. In a manner similar to section 2.4, we attach a homological local system $\underline{\tilde{\Lambda}}_{k-1}$ to $\tilde{\Lambda}_{k-1}$ by defining $\tilde{\Lambda}_{k-1}(\tau) = \tilde{\Lambda}^{U_\tau}_{k-1}$. We also attach a cohomological local system $\underline{\tilde{A}}^{k-1}$ to $\tilde{C}^{k-1}_{har}$ which is the dual one to $\underline{\tilde{\Lambda}}_{k-1}$.

Proposition 3.8:

1. *$\tilde{\Lambda}_{k-1}(\tau)$ is spanned by $\{\tilde{\lambda}_\sigma | \sigma \in \hat{\mathcal{T}}_{k-1}(\tau)\}$.*
2. *The complex $C_\bullet(\hat{\mathcal{T}}, \underline{\tilde{\Lambda}}_{k-1}) \to \tilde{\Lambda}_{k-1} \to 0$ is a projective resolution in the category of smooth G-modules.*
3. *The complex $0 \to \tilde{C}^{k-1}_{har} \to C^\bullet(\hat{\mathcal{T}}, \underline{\tilde{A}}^{k-1})$ is a resolution of G-modules.*
4. *For each k, r and $\tau \in \mathcal{T}_r$, there is an exact sequence*

$$0 \to \Lambda_k(\tau) \xrightarrow{\partial} \tilde{\Lambda}_{k-1}(\tau) \xrightarrow{\psi} \Lambda_{k-1}(\tau) \to 0 \tag{20}$$

giving rise to an exact sequence

$$0 \to C_r(\hat{\mathcal{T}}, \underline{\Lambda}_k) \xrightarrow{\partial} C_r(\hat{\mathcal{T}}, \underline{\tilde{\Lambda}}_{k-1}) \xrightarrow{\psi} C_r(\hat{\mathcal{T}}, \underline{\Lambda}_{k-1}) \to 0. \tag{21}$$

5. *Dually, one has the exact sequences*

$$0 \to A^{k-1}(\tau) \xrightarrow{\psi^*} \tilde{A}^{k-1}(\tau) \xrightarrow{\partial^*} A^k(\tau) \to 0,$$
$$0 \to C^r(\hat{\mathcal{T}}, \underline{A}^{k-1}) \xrightarrow{\psi^*} C^r(\hat{\mathcal{T}}, \underline{\tilde{A}}^{k-1}) \xrightarrow{\partial^*} C^r(\hat{\mathcal{T}}, \underline{A}^k) \to 0.$$

Proof:

1. Clearly, all the $\tilde{\lambda}_\sigma$ for $\sigma \in \hat{\mathcal{T}}_{k-1}(\tau)$ are fixed by U_τ (see (9)). For the other inclusion we use the exact sequence (8). Suppose that $f \in \tilde{\Lambda}_{k-1}$ is fixed by U_τ; then $\psi f \in \Lambda_{k-1}$ is also fixed by U_τ. Hence by Proposition 2.1, we can write $\psi f = \sum_{\sigma\in\hat{\mathcal{T}}_{k-1}(\tau)} a_\sigma \lambda_\sigma$. Let $g = \sum_{\sigma\in\hat{\mathcal{T}}_{k-1}(\tau)} a_\sigma \tilde{\lambda}_\sigma$; then $f - g \in ker(\psi)$, hence there exists $f' \in \Lambda_k$ such that $f - g = \partial(f')$. Since $f - g$ is fixed by U_τ, and ∂ is injective, f' is also fixed by U_τ, hence is a linear combination of $\lambda_{\sigma'}$ for $\sigma' \in \hat{\mathcal{T}}_k(\tau)$. The result now follows from (3).
2. Follows from 1., again by [S-S2], p. 29.
3. Since the exact sequence in 2. is a split exact sequence of free abelian groups, its F-dual is also exact.
4. For the first exact sequence, since the funcor of U_τ-invariants is left exact, we only have to prove surjectivity; but this follows from Lemma 3.3 and Proposition 2.1. The second assertion follows.
5. Follows exactly as in 3. ∎

Remark 3.9: As in Remark 2.4, we view an element of $C^r(\hat{\mathcal{T}}, \underline{\tilde{\mathcal{A}}}^k)$ as a function of two variables $(\sigma;\tau)$ via its values on the $\tilde{\lambda}_\sigma$ for $\sigma \in \hat{\mathcal{T}}_k(\tau)$.

3.4. A SPLITTING. For $\tau \in \hat{\mathcal{T}}_{k-1}$ we will now define a map $\rho_\tau \colon \tilde{\Lambda}_{k-1}(\tau) \to \Lambda_k(\tau)$ that will give a splitting of (20). This map depends, in a crucial way, on the choice of the leading vertex for τ.

First note that a lattice M gives rise to a lifting $n_M \colon \mathcal{A} \to \tilde{\mathcal{A}}$ assigning to an element of $\mathcal{A}$ its unique lifting in $M \setminus \pi M$. Now let M be the leading vertex of τ. Duplicate n_M to a map $n_M \colon \mathcal{A}^{k+1} \to \tilde{\mathcal{A}}^{k+1}$ and let $\rho_\tau = n_M^*$, i.e.

$$\rho_\tau(f)(a_0, \ldots, a_k) = f(n_M(a_0), \ldots, n_M(a_k)).$$

LEMMA 3.10: *We have, for $\sigma \in \hat{\mathcal{T}}_{k-1}(\tau)$,*

$$\rho_\tau(\tilde{\lambda}_\sigma) = \lambda_{M\sigma} \tag{22}$$

where $M\sigma$ is the simplex obtained from σ by adjoining to it M as a leading vertex. (Note that this is still a simplex. If M is already a vertex of σ, we view it as a degenerate simplex, and then the result is 0.)

Proof: Let $S = (a_0, \ldots, a_k)$ be an element of $\mathcal{A}^{k+1}$ and let $\tilde{S} = (\tilde{a}_0, \ldots, \tilde{a}_k)$ be a lifting of it in $\widetilde{\mathcal{A}}^{k+1}$. We have

$$\begin{aligned}\rho_\tau(\tilde{\lambda}_\sigma)(S) = \tilde{\lambda}_\sigma(n_M(a_0),\ldots,n_M(a_k)) &= -\sum_{i=0}^{k}(-1)^i(\sigma,S_i)ord_{L_0}(n_M(a_i))\\ &= -\sum_{i=0}^{k}(-1)^i(\sigma,S_i)(ord_{L_0}-ord_M)(\tilde{a}_i)\\ &=(M\sigma,S).\end{aligned}$$

The last equality follows from (4), since $(M\sigma)_0=\sigma$. ∎

PROPOSITION 3.11: *The map ρ_τ is a splitting of the exact sequence (20).*

Proof: Let us verify that $\rho_\tau\circ\partial = id$ on the generators of $\Lambda_k(\tau)$, namely λ_σ for $\sigma\in\hat{\mathcal{T}}_k(\tau)$:

$$\rho_\tau(\partial\lambda_\sigma)=\rho_\tau\left(\sum_{i=0}^{k}(-1)^i\tilde{\lambda}_{\sigma_i}\right)=\sum_{i=0}^{k}(-1)^i\lambda_{M\sigma_i}=\lambda_\sigma.$$

The last equality follows from (1). ∎

Remark 3.12: Although for $g\in G$, $\rho_{g\tau}=g\rho_\tau$, it is impossible to choose leading vertices for all $\tau\in\mathcal{T}_{k-1}$ in a G-equivariant way, and therefore it is impossible to glue the ρ_τs to get a G-splitting of (21).

3.5. RELATIONS TO KNOWN RESULTS FROM REPRESENTATION THEORY. As mentioned in the introduction, the extension (8), at least after tensored with $\mathbb{Q}$, is probably not new. What is new is the combinatorial interpretation, and the way we model it on harmonic cochains on the building. We now elaborate on this point.

Let $\mathbf{G}=PGL_{d+1}(K)$. For a subset $I\subseteq\{1,\ldots,d\}$ let P_I be the parabolic subgroup "with breaks in the complement of I". Thus $P=P_\emptyset$ is the Borel subgroup of $\mathbf{G}$ and if $I=\{1,\ldots,d-k\}$ our P_I is the group denoted by this symbol in [A-dS] (1.6). See also [S-S], Section 4. For any I and any ring R let

$$V_I(R)=C^\infty(\mathbf{G}/P_I,R)/\sum_{i\notin I}C^\infty(\mathbf{G}/P_{I\cup\{i\}},R).$$

The representation $V_I(\mathbb{Q})$ is known to be an irreducible admissible representation of $\mathbf{G}$, and in fact these are all the irreducible constituents of $C^\infty(\mathbf{G}/P,R)$. Furthermore, as explained in [A-dS], there is a canonical isomorphism

$$\Lambda_k\cong V_{I_k}(\mathbb{Z})$$

with $I_k = \{1, \dots, d-k\}$.

Now it follows from the work of Casselman that the smooth dual of $V_I(\mathbb{Q})$ is $V_I(\mathbb{Q})' = V_{\bar{I}}(\mathbb{Q})$ with $\bar{I} = \{d+1-i | i \in I\}$. Tensoring (8) with $\mathbb{Q}$ and taking smooth duals we therefore get a short exact sequence

$$0 \to V_{\bar{I}_{k-1}}(\mathbb{Q}) \to \tilde{\Lambda}_{k-1}(\mathbb{Q})' \to V_{\bar{I}_k}(\mathbb{Q}) \to 0.$$

Note that $\bar{I}_k = \{k+1, \dots, d\}$ and $\bar{I}_{k-1} = \{k, \dots, d\}$, and that $P_{\bar{I}_k} \subset P_{\bar{I}_{k-1}}$. There is an obvious extension

$$0 \to C^\infty(\mathbf{G}/P_{\bar{I}_{k-1}}, \mathbb{Q}) / \sum_{i \notin \bar{I}_{k-1}} C^\infty(\mathbf{G}/P_{\bar{I}_{k-1} \cup \{i\}}, \mathbb{Q}) \to E$$
$$\to C^\infty(\mathbf{G}/P_{\bar{I}_k}, \mathbb{Q}) / \sum_{i \notin \bar{I}_k} C^\infty(\mathbf{G}/P_{\bar{I}_k \cup \{i\}}, \mathbb{Q}) \to 0,$$

where

$$E = C^\infty(\mathbf{G}/P_{\bar{I}_k}, \mathbb{Q}) / \sum_{i \notin \bar{I}_{k-1}} C^\infty(\mathbf{G}/P_{\bar{I}_k \cup \{i\}}, \mathbb{Q}).$$

We have not checked it, but most likely the two extensions, $\tilde{\Lambda}_{k-1}(\mathbb{Q})'$ and E, coincide.

It is interesting to see what all this says for the case of the tree, $d = 1$, in particular with regard to integral structures. The last sequence, with $k = 1$, is simply

$$0 \to \mathbb{Q} \to C^\infty(\mathbf{G}/P, \mathbb{Q}) \to C^\infty(\mathbf{G}/P, \mathbb{Q})/\mathbb{Q} \to 0,$$

which defines the Steinberg representation. Since it must coincide with the smooth dual of (8), there must be a canonical isomorphism of $C^\infty(\mathbf{G}/P, \mathbb{Q})/\mathbb{Q}$ with the **smooth** harmonic 1-cochains. Now harmonic cochains correspond naturally to the (full) dual of $C^\infty(\mathbf{G}/P, \mathbb{Q})/\mathbb{Q}$, namely to distributions on $\mathbf{G}/P$ with total mass 0. The smooth ones are those that are invariant (Haar) with respect to some open subgroup of $\mathbf{G}$. To exhibit the isomorphism between the Steinberg representation and its smooth dual we have to construct a $\mathbf{G}$-equivariant non-degenerate pairing on $\Lambda_1(\mathbb{Q})$. This is done as follows. Let $\rho(\sigma, \tau)$, for two oriented edges σ and τ, denote the **maximal** distance between vertices of σ and τ. Let $sgn(\sigma, \tau)$ be 1 if they point in the same direction, and -1 if they point in opposite directions. Define

$$\langle \lambda_\sigma, \lambda_\tau \rangle = sgn(\sigma, \tau) q^{1-\rho(\sigma,\tau)}.$$

Then this is a well-defined $\mathbf{G}$-pairing on Λ_1, exhibiting it as its own smooth dual. Note that it is defined over $\mathbb{Q}$, but not over $\mathbb{Z}$.

Note also that in the category of smooth **G**-modules, although the representations at the two ends of (8) are, in the tree case, self-dual, this is not true for the extension itself, or else it would have split.

4. Application to the cohomology of discrete subgroups

Let Γ be a discrete cocompact subgroup of G. In this section we compute cohomology with coefficients in $\mathbb{Q}$-modules. In doing so we may pass to a subgroup $\Gamma' \triangleleft \Gamma$ of finite index, and then take Γ/Γ'-invariants. Therefore, we may assume, without loss of generality, that Γ is torsion free, and acts freely on $\hat{\mathcal{T}}$.

4.1. Garland's isomorphism. Let us briefly recall the isomorphism used by Garland in [?],

$$H^0(\Gamma, C^d_{har}(\mathbb{Q})) \simeq H^d(\Gamma, \mathbb{Q}).$$

Let $\hat{\mathcal{T}}_\Gamma = \hat{\mathcal{T}}/\Gamma$ and let $C^s(\hat{\mathcal{T}}_\Gamma) = C^s(\hat{\mathcal{T}}, \mathbb{Q})^\Gamma$ be the finite dimensional space of **$\mathbb{Q}$-valued** alternating s-cochains on $\hat{\mathcal{T}}_\Gamma$, endowed with the inner product

$$(f, g) = \sum_{\sigma \in \mathcal{T}_s/\Gamma} f(\sigma) g(\sigma).$$

(Note that the summand does not depend on the oriented representative of σ.) Let d: $C^s(\hat{\mathcal{T}}_\Gamma) \to C^{s+1}(\hat{\mathcal{T}}_\Gamma)$ be the usual coboundary map. Since $C_\bullet(\hat{\mathcal{T}}, \mathbb{Q}) \to \mathbb{Q} \to 0$ is a projective resolution of Γ-modules,

$$H^s(\Gamma, \mathbb{Q}) = h^s(C^\bullet(\hat{\mathcal{T}}_\Gamma), d). \tag{23}$$

It follows that

$$H^d(\Gamma, \mathbb{Q}) \simeq coker(d : C^{d-1}(\hat{\mathcal{T}}_\Gamma) \to C^d(\hat{\mathcal{T}}_\Gamma)).$$

Let δ: $C^d(\hat{\mathcal{T}}_\Gamma) \to C^{d-1}(\hat{\mathcal{T}}_\Gamma)$ be the adjoint map to d, that is $(x, dy) = (\delta x, y)$. Then $im(d)^\perp = \ker(\delta)$ and hence $coker(d) \simeq \ker(\delta)$.

Lemma 4.1: *We have* $\ker(\delta) = C^d_{har}(\mathbb{Q})^\Gamma$.

Proof: For an alternating cohain $c \in C^d(\hat{\mathcal{T}}_\Gamma)$, one has to show that the condition $\delta(c) = 0$ is equivalent to conditions (A)–(D) in Theorem 2.3.

First note that condition (A) says that the cochain is alternating, and that conditions (B) and (C) are void at the top degree d.

The map δ is given by

$$\delta(c)(\tau) = \sum_{\tau < \sigma} [\sigma : \tau] c(\sigma).$$

Here $\tau \in \hat{\mathcal{T}}_{d-1}$, the sum is over the $\sigma \in \mathcal{T}_d$ of which τ is a face, and each such σ is given an arbitrary orientation. In order to avoid the arbitrariness of the orientation we insist that the leading vertex of σ be the same as the one for τ. The condition $\delta(c) = 0$ comes out to be exactly condition (B). ∎

We conclude that

$$H^d(\Gamma, \mathbb{Q}) \simeq \ker(\delta) = H^0(\Gamma, C^d_{har}(\mathbb{Q})).$$

We shall refer to this isomorphism as **Garland's isomorphism**. Explicitly, a Γ-invariant d-cochain $f \in H^0(\Gamma, C^d_{har}(\mathbb{Q}))$ maps under this isomorphism to the cohomology class of f in $h^d(C^\bullet(\hat{\mathcal{T}}, \mathbb{Q})^\Gamma)$.

4.2. The connecting homomorphism. Let F be a field of characteristic 0, $1 \leq s \leq d$ and $r \geq 0$.

Definition 4.2: Let ν: $H^r(\Gamma, C^s_{har}) \to H^{r+1}(\Gamma, C^{s-1}_{har})$ be the connecting homomorphism coming from the extension (10).

By ([S-S], theorem 3, p. 93) we have

$$\dim H^r(\Gamma, C^s_{har}) = \begin{cases} \delta_{r,s}, & r+s \neq d, \\ \mu(\Gamma) + \delta_{r,s}, & r+s = d, \end{cases}$$

where $\mu(\Gamma) = \dim H^d(\Gamma, K) < \infty$. Thus, the only interesting case is when $r + s = d$.

Theorem 4.3: *Assume that $r + s = d$. Then ν is an isomorphism, except when $r = s - 2$, in which case ν is injective, or $r = s$, in which case it is surjective.*

Proof: By extension of scalars we may assume $F = \mathbb{Q}$.

The dimensions of the spaces $H^r(\Gamma, C^s_{har})$ being as they are, it suffices to prove that the composition

$$\nu^d \colon H^0(\Gamma, C^d_{har}) \to H^d(\Gamma, \mathbb{Q})$$

is an isomorphism. In fact, we shall prove that ν^d is equal, up to a constant sign, to Garland's isomorphism.

The cohomology spaces $H^r(\Gamma, C^s_{har})$ can be described via the explicit resolutions of Theorem 2.2 (The modules $C^r(\hat{\mathcal{T}}, \underline{A}^s)$ of these resolutions are Γ-acyclic because, at least when Γ is small enough, they are induced Γ-modules.) Thus

$$H^r(\Gamma, C^s_{har}) = h^r(C^\bullet(\hat{\mathcal{T}}, \underline{A}^s)^\Gamma).$$

Moreover, the connecting homomorphism can be computed via the the commutative exact diagram described in Proposition 3.8:

$$\begin{array}{ccccccccc}
 & & 0 & & 0 & & 0 & & \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^{s-1}_{har} & \longrightarrow & \widetilde{C}^{s-1}_{har} & \longrightarrow & C^{s}_{har} & \longrightarrow & 0 \\
 & & \downarrow & & \downarrow & & \downarrow & & \\
0 & \longrightarrow & C^\bullet(\hat{\mathcal{T}}, \underline{A}^{s-1}) & \xrightarrow{\psi^*} & C^\bullet(\hat{\mathcal{T}}, \underline{\widetilde{A}}^{s-1}) & \xrightarrow{\partial^*} & C^\bullet(\hat{\mathcal{T}}, \underline{A}^{s}) & \longrightarrow & 0.
\end{array}$$

Taking Γ-invariants, we get at the rth stage:

$$\begin{array}{ccccccccc}
0 & \longrightarrow & C^r(\hat{\mathcal{T}}, \underline{A}^{s-1})^\Gamma & \xrightarrow{\psi^*} & C^r(\hat{\mathcal{T}}, \underline{\widetilde{A}}^{s-1})^\Gamma & \xrightarrow{\partial^*} & C^r(\hat{\mathcal{T}}, \underline{A}^{s})^\Gamma & \longrightarrow & 0 \\
 & & \downarrow d & & \downarrow d & & \downarrow d & & \\
0 & \longrightarrow & C^{r+1}(\hat{\mathcal{T}}, \underline{A}^{s-1})^\Gamma & \xrightarrow{\psi^*} & C^{r+1}(\hat{\mathcal{T}}, \underline{\widetilde{A}}^{s-1})^\Gamma & \xrightarrow{\partial^*} & C^{r+1}(\hat{\mathcal{T}}, \underline{A}^{s})^\Gamma & \longrightarrow & 0.
\end{array}$$

Since $C^r(\hat{\mathcal{T}}, \underline{A}^{s-1})$ is Γ-acyclic, the rows of the above diagram are still exact. The connecting homomorphism ν is computed by chasing this diagram. An element of $H^r(\Gamma, C^s_{har})$ is represented by an element $x \in C^r(\hat{\mathcal{T}}, \underline{A}^s)^\Gamma$ annihilated by d. Lift it arbitrarily to $y \in C^r(\hat{\mathcal{T}}, \underline{\widetilde{A}}^{s-1})^\Gamma$. Since $\partial^* dy = 0$, there is a unique $z \in C^{r+1}(\hat{\mathcal{T}}, \underline{A}^{s-1})^\Gamma$ such that $\psi^* z = dy$. Then $dz = 0$, and $[z] = \nu[x]$.

To compute ν^d, we start with an element $f \in H^0(\Gamma, C^d_{har})$ and map it to $f_0 \in C^0(\hat{\mathcal{T}}, \underline{A}^d)^\Gamma$, via the embedding $H^0(\Gamma, C^d_{har}) \hookrightarrow C^0(\hat{\mathcal{T}}, \underline{A}^d)^\Gamma$.

If we can find elements $f_r \in C^r(\hat{\mathcal{T}}, \underline{A}^s)^\Gamma$ ($r = 1, \ldots, d$, $s = d - r$) and $\tilde{f}_r \in C^r(\hat{\mathcal{T}}, \underline{\widetilde{A}}^{s-1})^\Gamma$ ($r = 0, \ldots, d-1$) such that

$$\begin{array}{ccc}
 & & \tilde{f}_r \stackrel{\partial^*}{\longmapsto} f_r \\
 & & \big\downarrow d \\
f_{r+1} & \stackrel{\psi^*}{\longmapsto} & d\tilde{f}_r
\end{array} \tag{24}$$

then one has $\nu[f_r] = [f_{r+1}]$ and hence $\nu^d f = [f_d]$.

Following Remarks 2.4 and 3.9, we view f_r and $\tilde{f}_r$ as functions of two variables $(\tau; \sigma)$. Using this point of view we will give explicitly such functions, satisfying (24). All of them will be defined simultaneously from f. Checking (24) will then be relatively easy. The difficulty will be in showing that these functions actually represent elements of $C^r(\hat{\mathcal{T}}, \underline{A}^s)^\Gamma$ and $C^r(\hat{\mathcal{T}}, \underline{\tilde{A}}^{s-1})^\Gamma$. This is where the local splittings ρ_τ will help us.

First we have, for $\tau \in \hat{\mathcal{T}}_0$ and $\sigma \in \hat{\mathcal{T}}_d(\tau)$,

$$f_0(\tau;\sigma) = f(\sigma),$$

since the embedding $H^0(\Gamma, C^d_{har}) \hookrightarrow C^0(\hat{\mathcal{T}}, \underline{A}^d)^\Gamma$ is the dual of the augmentation map $\bigoplus \Lambda_d^{U_\tau} \to \Lambda_d$ under which, for $\sigma \in \hat{\mathcal{T}}_0(\tau)$, $\lambda_\sigma \in \Lambda_d^{U_\tau}$ maps to itself.

To introduce $\tilde{f}_r$, we look for a doubly alternating function on $(\tau;\sigma)$ where $\tau \in \hat{\mathcal{T}}_r$, $\sigma \in \hat{\mathcal{T}}_{s-1}(\tau)$ and $r+s=d$. To this end note that if σ and τ have no vertex in common, then they can be interlaced to form a simplex of $\hat{\mathcal{T}}_d$.

Assume that $\tau = (u_0,\ldots,u_r)$, $\sigma = (v_0,\ldots,v_{s-1})$, and that $v = (w_0,\ldots,w_d)$ satisfies $v = \tau \cup \sigma$ as a set of vertices. Define

$$\binom{\tau,\sigma}{v} = sgn\begin{pmatrix} u_0u_1\cdots u_r v_0 v_1 \cdots v_{s-1} \\ w_0 w_1 \cdots w_d \end{pmatrix}$$

where the parentheses mean the permutation moving the objects in the upper row to those in the lower row. We are now ready to define, for $\tau \in \hat{\mathcal{T}}_r$ and $\sigma \in \hat{\mathcal{T}}_{s-1}(\tau)$,

$$\tilde{f}_r(\tau;\sigma) = \begin{cases} (-1)^{1+2+\cdots+r}\binom{\tau,\sigma}{\tau\cup\sigma} f(\tau\cup\sigma) & |\sigma\cap\tau| = 0, \\ 0 & |\sigma\cap\tau| \neq 0, \end{cases} \tag{25}$$

where the leading vertex of $\tau\cup\sigma$ is chosen arbitrarily. Note that since f is an alternating function, $\tilde{f}_r(\tau;\sigma)$ does not depend on this choice. Define also for $r \geq 1$, $\tau \in \hat{\mathcal{T}}_r$ and $\sigma \in \hat{\mathcal{T}}_s(\tau)$,

$$f_r(\tau;\sigma) = \sum_{i=0}^{r}(-1)^i \tilde{f}_{r-1}(\tau_i;\sigma). \tag{26}$$

To see that the f_r and $\tilde{f}_r$ satisfy (24), let us first write explicitly the maps d, ∂^* and ψ^* on functions: One has, by (3),

$$\partial^* f(\tau;\sigma) = [\partial^* f(\tau)](\lambda_\sigma) = f(\tau)(\partial\lambda_\sigma) = \sum_i (-1)^i f(\tau)(\tilde{\lambda}_{\sigma_i}) = \sum_i (-1)^i f(\tau;\sigma_i)$$

and, by (6),

$$\psi^* f(\tau;\sigma) = f(\tau)(\psi\tilde{\lambda}_\sigma) = f(\tau)(\lambda_\sigma) = f(\tau;\sigma).$$

One also has, by definition,

$$df(\tau;\sigma) = \sum(-1)^i f(\tau_i;\sigma).$$

The last formula and (26) show that $d\tilde{f}_r = \psi^* f_{r+1}$. To show that (24) holds, we still have to check that $\partial^* \tilde{f}_r = f_r$. Since ψ^* is injective, it is enough to prove that

$$\psi^* \partial^* \tilde{f}_r = d\tilde{f}_{r-1}. \tag{27}$$

Let $\tau = (u_0, \ldots, u_r) \in \hat{\mathcal{T}}_r$ and $\sigma = (v_0, \ldots, v_s) \in \hat{\mathcal{T}}_s(\tau)$. Since $r+s = d$, they must have at least one vertex in common. If they only have a single vertex $u_i = v_j$ in common, and $\tau \cup \sigma = (w_0, \ldots, w_d)$, then

$$\psi^* \partial^* \tilde{f}_r(\tau;\sigma) = \partial^* \tilde{f}_r(\tau;\sigma) = (-1)^j \tilde{f}_r(\tau, \sigma_j) = (-1)^j (-1)^{1+\cdots+r} \begin{pmatrix} \tau, \sigma_j \\ \tau \cup \sigma \end{pmatrix} f(\tau \cup \sigma)$$

and

$$d\tilde{f}_{r-1}(\tau;\sigma) = (-1)^i \tilde{f}_{r-1}(\tau_i;\sigma) = (-1)^i (-1)^{1+\cdots+(r-1)} \begin{pmatrix} \tau_i, \sigma \\ \tau \cup \sigma \end{pmatrix} f(\tau \cup \sigma).$$

One has only to verify that

$$(-1)^j (-1)^r \begin{pmatrix} \tau, \sigma_j \\ \tau \cup \sigma \end{pmatrix} = (-1)^i \begin{pmatrix} \tau_i, \sigma \\ \tau \cup \sigma \end{pmatrix}.$$

This is equivalent to

$$(-1)^{i+j+r} sgn \begin{pmatrix} u_0 \cdots u_i \cdots u_r v_0 \cdots \widehat{v_j} \cdots v_s \\ w_0 \cdots w_d \end{pmatrix} = sgn \begin{pmatrix} u_0 \cdots \widehat{u_i} \cdots u_r v_0 \cdots v_j \cdots v_s \\ w_0 \cdots w_d \end{pmatrix}$$

which is clear. If $|\tau \cap \sigma| > 1$ then both sides of (27) are clearly zero.

Special care must be given to the case $r = 0$ where the definition of f_0 is special: we have to check that $f_0 = \partial^* \tilde{f}_0$. Indeed, if $\sigma = (v_0, \ldots, v_d)$ and $\tau = v_i$, then since $\sigma \cup \tau = \sigma$,

$$\partial^* \tilde{f}_0(\tau;\sigma) = (-1)^i \tilde{f}_0(\tau;\sigma_i) = (-1)^i \begin{pmatrix} \tau, \sigma_i \\ \sigma \end{pmatrix} f(\sigma) = f(\sigma) = f_0(\tau;\sigma).$$

Our next step will be to show that the functions f_r and $\tilde{f}_r$ actually represent elements of $C^r(\hat{\mathcal{T}}, \underline{A}^s)$ and $C^r(\hat{\mathcal{T}}, \underline{\tilde{A}}^{s-1})$. For a given τ, one has to show that there exist functionals $f_r(\tau) \in \Lambda_s(\tau)^*$ and $\tilde{f}_r(\tau) \in \tilde{\Lambda}_{s-1}(\tau)^*$ such that

$$f_r(\tau;\sigma) = f_r(\tau)(\lambda_\sigma), \quad \tilde{f}_r(\tau;\sigma) = \tilde{f}_r(\tau)(\tilde{\lambda}_\sigma).$$

We will do it by induction on r: Assume that either $r = 0$, or $r \geq 1$ and the claim is true for $r-1$. Then $f_r \in C^r(\hat{\mathcal{T}}, \underline{A}^s)$ because, if $r \geq 1$, $\psi^* f_r = d\tilde{f}_{r-1}$, and for $r=0$, f_0 was directly defined as an element of $C^0(\hat{\mathcal{T}},\underline{A}^d)$. Thus, the desired $f_r(\tau)$ exist. We will now prove that for any $r\geq 0$,

$$\tilde{f}_r(\tau;\sigma)=\rho_\tau^* f_r(\tau)(\tilde{\lambda}_\sigma).$$

Indeed, denoting by M the leading vertex of τ, we have by (22), for $\sigma\in\hat{\mathcal{T}}_{s-1}(\tau)$,

$$\begin{aligned}\rho_\tau^* f_r(\tau)(\tilde{\lambda}_\sigma)=&f_r(\tau)(\rho_\tau\tilde{\lambda}_\sigma)=f_r(\tau)(\lambda_{M\sigma})\\ =&f_r(\tau,M\sigma)=d\tilde{f}_{r-1}(\tau,M\sigma)=\tilde{f}_{r-1}(\tau_0,M\sigma)\\ =&(-1)^{1+\cdots+(r-1)}\binom{\tau_0,M\sigma}{\tau\cup\sigma}f(\tau\cup\sigma)\\ =&(-1)^{1+\cdots+r}\binom{\tau,\sigma}{\tau\cup\sigma}f(\tau\cup\sigma)=\tilde{f}_r(\tau;\sigma).\end{aligned}$$

The Γ-invariance of f_r and $\tilde{f}_r$ is clear since for $\gamma\in\Gamma$, f is γ-invariant and one has

$$\binom{\tau,\sigma}{v}=\binom{\gamma\tau,\gamma\sigma}{\gamma v}.$$

Our check that the f_r and $\tilde{f}_r$ are valid representatives is now complete. We have $\nu^d f=[f_d]$, and for $\tau=(u_0,\ldots,u_d)$, $\sigma=(u_j)$, and $\tau\cup\sigma=\tau$,

$$\begin{aligned}f_d(\tau;\sigma)&=d\tilde{f}_{d-1}(\tau;\sigma)=(-1)^j\tilde{f}_{d-1}(\tau_j;\sigma)\\ &=(-1)^j\binom{\tau_j,\sigma}{\tau}(-1)^{1+\cdots+(d-1)}f(\tau)=(-1)^{1+\cdots+d}f(\tau).\end{aligned}$$

This shows that, up to a sign, the harmonic Γ-invariant d-cochain f represents the cohomology class $[f_d]$ in

$$H^d(\Gamma,\mathbb{Q})=h^d(C^\bullet(\hat{\mathcal{T}},\mathbb{Q})^\Gamma),$$

hence ν^d is equal, up to a sign, to Garland's isomorphism. ∎

COROLLARY 4.4: *The extension* (10) *is non-split.*

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